International Baccalaureate

Extended Essay

Research Question:
What is the relationship between the Mandelbrot set and the Julia set; and how can these sets be expressed mathematically?

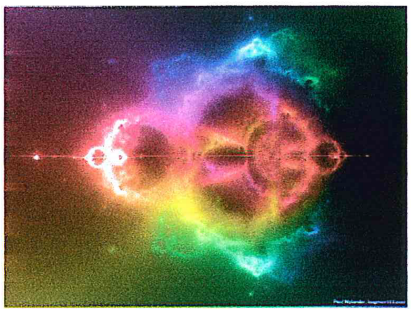
Topic: Fractals

Subject: Mathematics

Word Count: 2639

Content

Introduction



My interest in fractals came from a class I had in Theory of Knowledge where I was supposed to create a presentation about art. I am not too interested in art, but I am however fondly interested in mathematics so I started looking for a connection between art and mathematics. I looked into how mathematical functions can generate patterns that could be classified as art and I found that the topic of fractal geometry fit this quite well. This topic showed a very good connection between arts and mathematics and I therefore created a presentation about fractals. While researching fractals, I soon realized that there is a lot of math behind it all that cannot be explained in a Theory of Knowledge presentation and this is why I decided I would write an extended essay about fractals.

Fractal geometry is a part of Chaos theory that gives us a view of the world which allows us to analyze complex forms of objects and textures that can be computer generated and found in nature.[1] A fractal can be described as a "*unique, digital art form, using mathematical formulas to create art with an infinite diversity of form, detail, colour and light.*"[2] It is defined as a geometric or physical structure that is iterated at even smaller or larger scales to produce a self-similar shape and each part of which has the same statistical character as the whole.[3]

As described in the explanation above, fractals are iterating patterns. An iteration is simply a repetition of the same process. In mathematical terms, if a function is executed by giving it a domain, it will give a range. This range is then used and but back into the function so it can again be executed, however giving a different range.

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[1] (Massacusettes, n.d.)
[2] (Harrington, n.d.)

[3] (Harrington, n.d.), (Fractal, n.d.)
3

An example of this can be shown by using the following function: $b_{n+1} = a + b_n$. To start the iteration, a value of a and b has to be assigned to the letters on the right hand side of the equation. If $a = 1$ and $b = 2$, then the following iteration would occur:

$1 + 2 = 3 \rightarrow$ The range is 3, while the domain is 2 (and 1 is the constant).

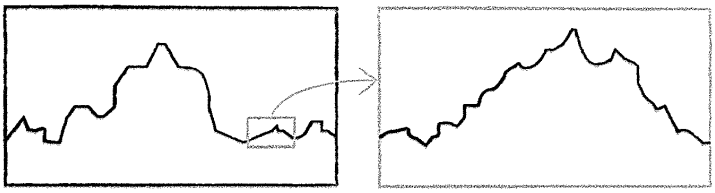
$1 + 3 = 4 \rightarrow$ The b place that was previous 2 has been substituted with the range of the previous function.

The ranges given by the iterating functions give us a sequence of numbers. The above example generated the short sequence 3, 4. If these numbers were complex, these points could be graphed. For iterating functions that can be plotted can be grouped into three categories:

- Convergence: the sequence of points converges to a limit
- Periodic cycle: for some value of the function, the sequence repeats itself (much similar to the real plane version of the Mandelbrot set)
- Chaos: The points go from one place to another in apparently chaotic manner

These categories will be revisited further into the essay.

There are multiple places where you can find fractals, one of them being the stock market. A stock market data graph can be an example of a fractal, as the graphs look similar at any scale[5].



[4] (Bogomoln, n.d.)
[5] (Shiffman, n.d.)
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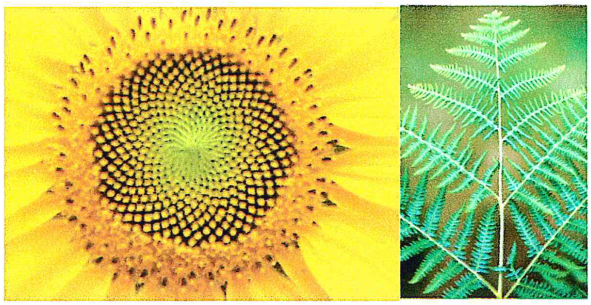
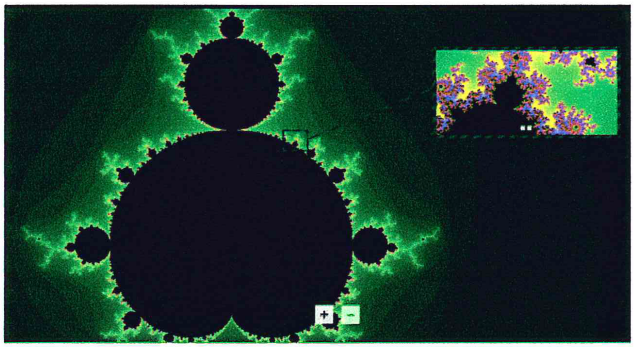
There are certain plants that resemble the form of a fractal, such as bracken or sunflowers. Nature creates magnificent plants and trees that resembles the form of a fractal.



Fractals must have become a big focus in mathematics when the first computer was developed because a computer has the ability to generate patterns and simulations with a function. Certain websites have even developed simulators that allows you to explore specific fractals. One website allowing this is the '*www.neave.com/fractal*'[6] and it simulates the Mandelbrot set. A picture of this type of zooming:



Benoit Mandelbrot was a mathematician that rediscovered fractals in the late 1970s because of computers[7]. This resulted in the revival of the fractal works of Gaston Julia and Pierre Fatou who were both French mathematicians who did their work in the 1900s and developed the idea of a fractal texture or object. Pierre Fatou was the first to introduce and study the Julia set and his contribution to this field in mathematics has had a large influence in the development of analysis due to the chaotic function of the Julia set. Do note that the works of

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[6] (Neave, n.d.)
[7] (Lesmoir-Gordon, 2010)
5

Pierre Fatou will not be discussed to a great extent in this extended essay as Gaston Julia's work completed Fatou's work. Gaston Julia's work gives a more extensive and complete explanation of the Julia set.[8]

The interconnection between the works of Pierre, Julia, and Mandelbrot will be described in this essay, however it is important to understand the mathematics behind the Mandelbrot set and the Julia set before doing so. Therefore the aim of this extended essay is to explore the Mandelbrot set and the Julia set and explain them in order to come to an understanding of how these sets behave as an iterating function to create a fractal and how these two sets are related.

Hence, my research question is as follows: **What is the relationship between the Mandelbrot set and the Julia set; and how can these sets be expressed mathematically?**

[8] (Robertson & O'Connor, School of Mathematics and Statistics, 2008) (Robertson & O'Connor, School of Mathematics and Statistics, 2008)

In some explanations a complex number is denoted by Z, an imaginary number is denoted by i, a complex value is denoted by c, 'all real numbers' set is denoted by $\mathbb{R}$, a quadratic polynomial is denoted by $P_{\mathbb{R}}$, a complex quadratic polynomial is denoted by P_c, the Mandelbrot set is denoted by M and the Julia set is denoted by J.

The sets that are described in this extended essay are sets that are simulated by a computer by plotting millions of points in order to create a clear image, however this will not be simulated in this extended essay. This is because that would require a computer coded program that iterates a function millions of times and this is beyond the comprehension of this paper.

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The Mathematics behind the Mandelbrot Set

In order to approach my research question, I am going to look in detail first at the Mandelbrot set, then the Julia set and finally I will look at the relationship between the two. The Mandelbrot set iterates and plots points in the complex plane, which resembles the shape and functions like a fractal.

The image shown in the first page of the introduction is a color coded image of the Mandelbrot set. The mathematics behind this is described below.

The Mandelbrot set is defined by a complex quadratic polynomial whereas the function for P_c is as follows: $P_c: c \rightarrow c$. This complex quadratic polynomial is what tells what is going on. It tells that the value one get in the function will be put back into it, which allows the function to iterate (as explained earlier).

The Mandelbrot set has a set of values of c in the complex plane given by the complex function: $Z_{n+1} = Z_n^2 + c$. Where n is the term or iteration the complex function starts at (setting n to infinite creates the Mandelbrot set as simulated on computers), c is a complex value, Z is the point where the set starts from in the complex plane.

By inserting the complex function into the complex quadratic polynomial, one gets the following: $P_c: Z \rightarrow Z^2 + c$. The Mandelbrot set can, however only exist under certain conditions. The conditions are the following: $M(c) \leftarrow (P_c^n(0)| \leq 2$ for all $n \geq 0)$. As long as the complex quadratic polynomial is kept between or *on* 2 and 0, the Mandelbrot set is true. However, if P_c allows for values that are greater than 2, the set will escape to infinity. This gives the base of what the Mandelbrot set does.

Because of the conditions of the Mandelbrot set, the range or the output that gives the sequence of the iterations will never be greater than 2 in both the negative plane and the positive plane in a complex plane.[9] Despite these conditions for the function of the Mandelbrot set, a similar function can be created to generate iterations that uses a number in the set $\mathbb{R}$ to get a better understanding of how it works. Configuring the complex quadratic polynomial (of the Mandelbrot set) into a quadratic polynomial by setting the complex number Z to the point where the set starts in the real plane, which is denoted by r; and setting the complex value c to $\mathbb{R}$ any real number, denoted by a. This will result in the following:

$$P: r \rightarrow r^2 + a.$$

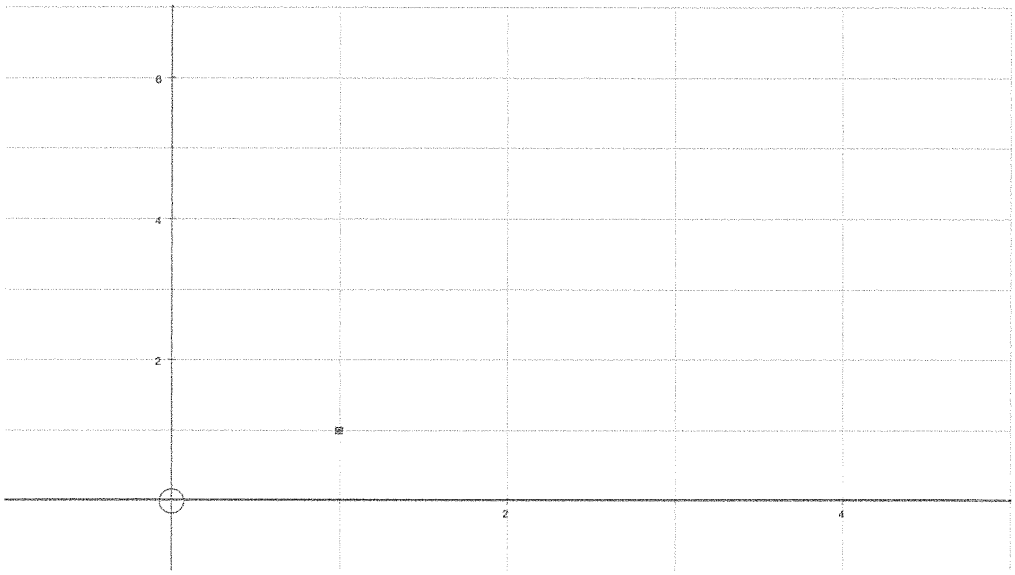
[9] (Ask a nerd: Mandelbrot, n.d.)

$P_{\mathbb{R}}$ can be initialized by setting a to any value, but in this case, the value will be set as $a = 1$. As r is the point where the function starts, setting $r = 0$ starts the function at the origin in the real plane.

Doing a single iteration of this gives the following:

$n = 1 \rightarrow 0^2 + 1 = 1$

This can be plotted as f(x) = x, x = 1 in the real plane:



Initializing $P_{\mathbb{R}}$ as an iterating function generates the following:

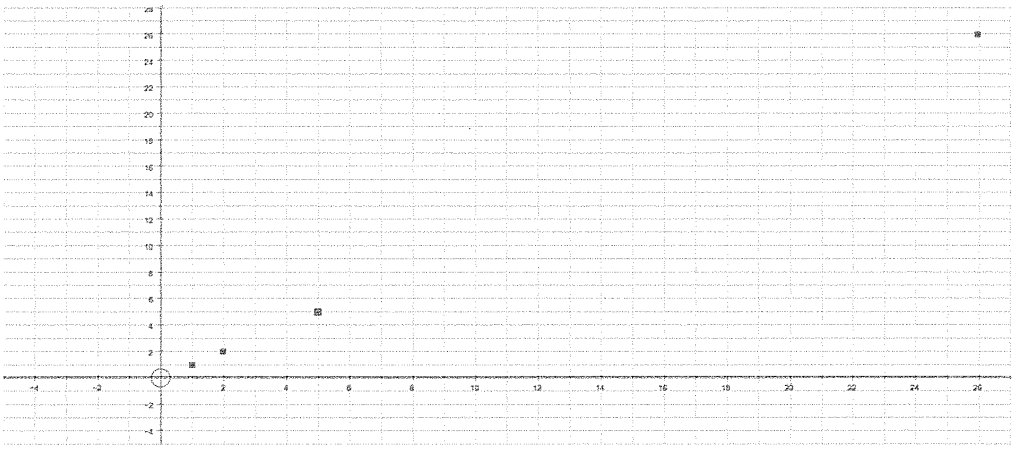
$n = 1 \rightarrow 0^2 + 1 = 1$

$n = 2 \rightarrow 1^2 + 1 = 2$

$n = 3 \rightarrow 2^2 + 1 = 5$

$n = 4 \rightarrow 5^2 + 1 = 26$

The iterations above are plotted in the real plane:



Another example of the Mandelbrot set in the real plane, is if $a = -1$. This gives us a bounded iteration. an iteration that repeats the same repeatedly and does not escape a specific boundary, which in this case is above 0 or below -1.

$n = 1 \rightarrow 0^2 - 1 = -1$

$n = 2 \rightarrow -1^2 - 1 = 0$

$n = 3 \rightarrow 0^2 - 1 = -1$

This gives a bounded iteration that generates the sequence -1, 0, -1, 0 to infinity.

If $c = -1$, or $c = 0$, then a bounded iteration will occur. This is an example of how the Mandelbrot set will generate an iteration that will be kept within a boundary.

The calculations above gives us an example of how the Mandelbrot set works with real numbers.

As the Mandelbrot set is actually drawn in the complex plane, complex numbers has to be used. By using the function previously described ($P_c: Z \rightarrow Z^2 + c$) for the Mandelbrot set. we can insert a complex number for c.

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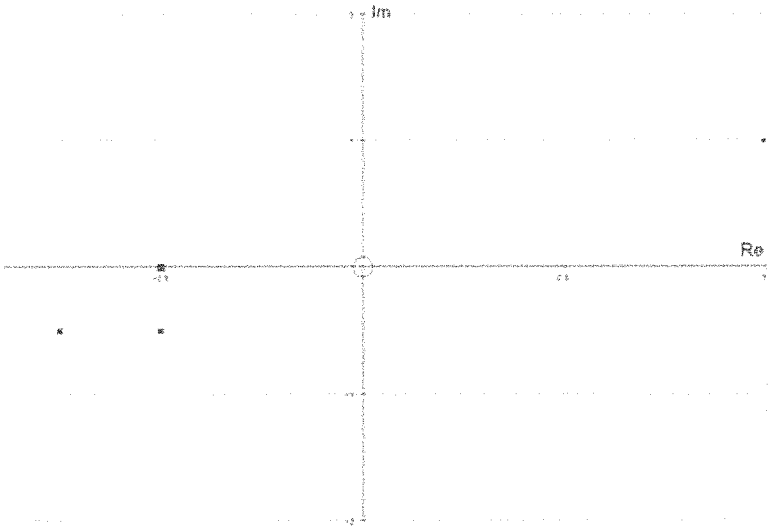
Like previously, the function will start at $Z = 0$, however a value for c has to be picked. As it is a complex number, it has to be something such as -0.5 - 0.5i. By initializing the function with this value gives the following iteration:

$n = 1 \rightarrow 0^2 - 0.5 - 0.5i = -0.5 - 0.5i$

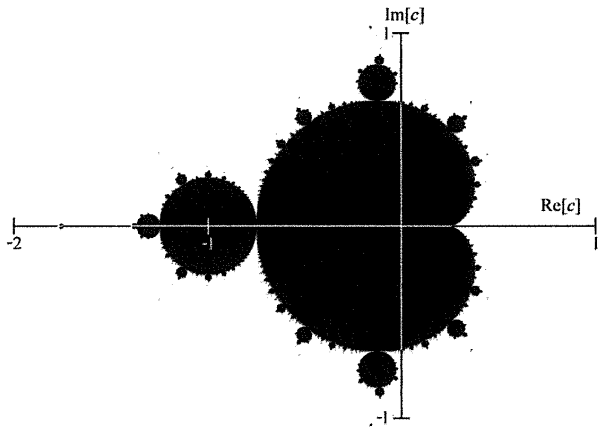
$n = 2 \rightarrow (-0.5 - 0.5i)^2 - 0.5 - 0.5i = -0.5$

$n = 3 \rightarrow (-0.5)^2 - 0.5 - 0.5i = -0.75 - 0.5i$

By plotting this iteration on the complex plane (as shown in the graph below), the start of the Mandelbrot set fractal is formed. By following the rule of the Mandelbrot set, any value below -2 and above 2, will not be counted for and therefore not plotted. On a computer, a function can be created in order to disallow the set to leave the boundary ($P_c^n(0)| \leq 2$ for all $n \geq 0$).



In total, by having a computer to iterate the function thousands of times, the Mandelbrot set looks like this:



The label "Im" refers to the imaginary plane while the label "Re" refers to the real plane.

Note: As described after the introduction, iterating the set manually would be extremely tedious and therefore an image of what the computer generated is shown above. Each black dot represents a coordinate in the complex plane, and to generate an image such as this manually would take a lot of time.

The Mandelbrot set can be put into two categories regarding iterations that were described in the introduction. Certain values for c makes $M(c)$ bounded, thus having a repeating pattern; hence it can be categorized as a periodic cycle. As $M(c)$ has boundaries to create a limit, the set also goes under the category of convergence.

The Mathematics behind the Julia Set

The complexity of the mathematics behind the Julia set goes beyond the sort of mathematical comprehension for this extended essay and it will therefore not be described to its full complexity.

Similarly to the Mandelbrot set, the Julia set is a set that generates points on the complex plane. The set is not as clearly defined and does not have a specific base function like the Mandelbrot set. The Julia set is rather a set that can be found within the Mandelbrot set and it is described by Encyclopedia of Mathematics as a compliment to the Fatou set.[10]

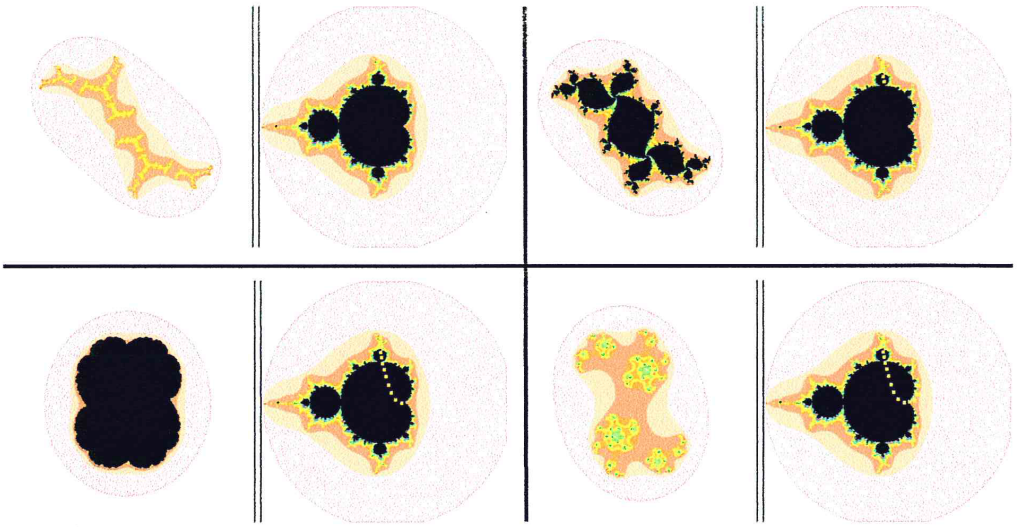
There is however one big difference between the two sets. The Julia set acts as a chaotic set; if a value is slightly altered, the set changes drastically and hence its categorization.[11]

As stated earlier, the Julia set is a chaotic set and it means that it is in the chaos category described before in *The Mathematics behind the Mandelbrot Set*. The Julia set had been unknown for the mainstream mathematical society for a long time and according to cut-the-knot.org, *"B. Mandelbrot has the following to say on the development of the theory, 'The resulting revival makes the properties of iterations essential for the theory of fractals. The fact that the Fatou-Julia findings did* ***not*** *develop to become the* ***source*** *of this theory suggests that even classical analysis needs intuition to develop, and can be helped by the computer. '"*

The Relationship between the Mandelbrot set and the Julia set

For the complex quadratic polynomial $M(P_c)$, in every value for c there exists a Julia set.

The "Yale classes" website[12] shows how the Julia set acts on a complex plane when given different values of c in its function. The following 4 pictures are depicted from a playing animation of the Julia set with different values for c.



TOP LEFT TO BOTTOM RIGHT

The yellow dots represent positions for c in the Mandelbrot set (right hand side of the double edged line) and it moves in an elliptical circle to show how the Julia set (left hand side of the double edged line) varies with the different points for c. Furthermore it shows how the Julia set moves from being a cluster of points whenever the values of c are inside the Mandelbrot set, however once the values leave the Mandelbrot set it disintegrates into dust or almost even a speck of points.

What can be seen from these pictures is that the Julia set acts dynamic; it keeps changing when different values for c are assigned. Due to the function of the Mandelbrot set, the set does not change the same way the Julia set does. It can be described as a static set; the Mandelbrot set takes on the characteristics of a convergence. As described in *The Mathematics behind the Mandelbrot Set*, iterating the Mandelbrot set in the real plane as well

[12] (Yale Classes, n.d.)

as in the complex plane, you can enter specific values of c that makes the function periodic. having the function output the same values repeatedly; thus making it a periodic cycle as well. These differences would show that the sets are quite different, but the connection between them makes them quite similar as well.

Conclusion

Coming back to the research question: What is the relationship between the Mandelbrot set and the Julia set; and how can these sets be expressed mathematically?

The Mandelbrot set and the Julia set can be expressed mathematically by defining the Mandelbrot set by P_c whereas the function for P_c is $P_c: C \rightarrow C$ and has a set of values in the complex plane. The values are given by the function $Z_{n+1} = Z_n^2 + c$ while also keeping within the boundary $(P_c^n(0)| \leq 2$ for all $n \geq 0)$. The Mandelbrot set acts static and is categorized as a converging iterating function while the Julia set is defined as a chaotic iterating function and acts dynamically.

Overall, the development of computers made it easier to work with the Fractal nature of some mathematical functions. This caused the revival of Gaston Julia's and Pierre Fatou's work by Mandelbrot. Mandelbrot used these sets to create his own, the Mandelbrot set; hence why there is a relation between the Mandelbrot set and the Julia set.

For the relationship between the Mandelbrot set and the Julia set it can be concluded that for every complex number c in $M(c)$ there exists a Julia set. For values of c within $M(c)$, the Julia set has the look of a cluster when plotted; however for values of c outside $M(c)$, the Julia set has the look of separated dust.

Bibliography

Ask a nerd: Mandelbrot. (n.d.). Retrieved August 27, 2013, from Bowdoing College: http://www.bowdoin.edu/~dfrancis/askanerd/mandelbrot/

Bogomoln, A. (n.d.). *Julia*. Retrieved August 29, 2013, from Cut the Knot: http://www.cut-the-knot.org/blue/julia.shtml

Encyclopedia of Mathematics. (n.d.). Retrieved October 8, 2013, from Julia Set: http://www.encyclopediaofmath.org/index.php?title=Julia_set&oldid=14959

Fractal. (n.d.). (Oxford Dictionaries) Retrieved August 21, 2013, from Oxford Dictionaries: http://oxforddictionaries.com/definition/english/fractal?q=fractal

Harrington, D. (n.d.). *What Are Fractals?* Retrieved August 21, 2013, from Fractal Arts: http://fractalarts.com/SFDA/whatarefractals.html

Holme, A. (1996). 11.4 Iterasjon, Julia mengder og Mandelbrot mengden. In A. Holme, *Fraktal Geometri* (pp. 153-155). Alma Mater Forlag AS.

Lesmoir-Gordon. N. (2010, October 17). *The Guardian*. (Guardian News and Media Limited) Retriev October 9, 2013, from The Guardian - Science - Benoit Mandelbrot Obituary: http://www.theguardian.com/science/2010/oct/17/benoit-mandelbrot-obituary

Log, S. (2005). Complex Number Theorems. In S. Log, *Mathema* (pp. 175-188). Tapir Akademisk Forlag.

Massacusettes, U. o. (n.d.). *Exploring Fractals, Intro*. Retrieved August 25, 2013, from Department Mathematics and Statistics: https://www.math.umass.edu/~mconnors/fractal/intro.html

Neave, P. (n.d.). *Neave*. Retrieved October 6, 2013, from Neave Fractal Zoomer: http://www.neave.com/fractal/

Peterson, I. (1992). *Kaos Fraktaler og Sæbebobler.* Borgens Forlag.

Robertson, E. F., & O'Connor, J. J. (2000, May). *School of Mathematics and Statistics*. Retrieved October 9, 2013, from Pierre Joseph Louis Fatou: http://www-history.mcs.st-andrews.ac.uk/Biographies/Fatou.html

Robertson, E. F., & O'Connor, J. J. (2008, December). *School of Mathematics and Statistics*. Retrieve October 9, 2013, from Gaston Maurice Julia: http://www-history.mcs.st-andrews.ac.uk/Biographies/Julia.html

Shiffman, D. (n.d.). *Chapter 8, Fractals*. Retrieved August 25, 2013, from Nature of Code: http://natureofcode.com/book/chapter-8-fractals/

Vignette 15: Julia Sets. (n.d.). (John Carroll University) Retrieved August 29, 2013, from http://www.jcu.edu/math/vignettes/Julia.htm

Yale Classes. (n.d.). Retrieved October 8, 2013, from The Mandelbrot Set - Relations with Julia Sets: http://classes.yale.edu/fractals/Mandelset/MandelDef/MandelJuliaRel/MandelJuliaRel.htr #JDAnchor

Mathematical ideas and calculations used in this Extended Essay have been developed from the Norwegian book *Innføring I Geometri* (Holme, 1996) and the Danish book *Kaos Fraktaler og Sæbebobler (Peterson, 1992)*.